A discrete quantum model of the harmonic oscillator

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Abstract
We construct a new model of the quantum oscillator, whose energy spectrum is equally-spaced and lower-bound, whereas the spectra of position and of momentum are a denumerable non-degenerate set of points in $[-1, 1]$ that depends on the deformation parameter $q \in (0, 1)$. We provide its explicit wavefunctions, both in position and momentum representations, in terms of the discrete q-Hermite polynomials. We build a Hilbert space with a unique measure, where an analogue of the fractional Fourier transform is defined in order to govern the time evolution of this discrete oscillator. In the limit when $q \to 1^-$, one recovers the ordinary quantum harmonic oscillator.

1. Introduction

Several algebraic constructions have been proposed in the literature to describe various extensions of the quantum harmonic oscillator. These constructions are based on various deformations of the standard oscillator Lie algebra, or different associative algebras. In most of these models, it is difficult to construct a theory for such oscillators, which is as complete as the well-known treatment of the standard harmonic oscillator in quantum mechanics; namely, a canonical complementarity between position and momentum, explicit forms for the wavefunctions, and a coherent description of time evolution.

The earliest model, generally called the q-oscillator, was proposed by Macfarlane [1] and Biedenharn [2] on the basis of raising and lowering eigenstates of a Hamiltonian with the q-deformed commutator $a_q^+ a_q - q a_q a_q^+$. A theory of this oscillator has been elaborated (see, e.g., [3–6]); yet, it has not been clear how to construct position and momentum operators satisfying the basic commutation relations with a Hamiltonian to characterize infinitesimal

harmonic motion. This may be one of the reasons why this *q*-oscillator has not proven attractive for many physicists.

The postulates we use to define oscillator models are the following [7, 8].

(1) There exists an essentially self-adjoint *position* operator Q, whose spectrum $\mathcal{X}$ is the set of positions $\{x\}$ of the system.
(2) There exists a self-adjoint and compact *Hamiltonian* operator H, whose commutator with position defines the momentum operator P,

$$[H, Q] =: -\mathrm{i}P, \tag{1}$$

and corresponds to the first Hamilton equation ($\mathrm{i} = \sqrt{-1}$); the commutator of the Hamiltonian with momentum returns the position operator

$$[H, P] = \mathrm{i}Q, \tag{2}$$

that corresponds with the second Hamilton equation, and which characterizes the oscillator dynamics. Equivalent to (1)–(2), one can propose the Newton–Lie equation as $[H, [H, Q]] = Q$. The set of momentum values of the system is the spectrum of P, which is equal to that of Q because (1) and (2) generate a rotation between these two operators (to be written below).
(3) The three operators, Q, P and H, close into an *associative algebra*, i.e., they satisfy the Jacobi identity,

$$[P, [H, Q]] + [Q, [P, H]] + [H, [Q, P]] = 0. \tag{3}$$

We note that the basic commutator $[Q, P]$ has not been defined. Due to (1) and (2), the only restriction imposed by the associativity condition (3), since the first two summands are identically zero, is that $[Q, P]$ must commute with H, and thus be constant under the oscillator motion. This indicates that each distinct choice of the basic commutator $[Q, P]$ will yield a distinct model for the oscillator. If the choice is the Heisenberg commutator $[Q, P] = \hbar\hat{1}$, one has the standard four-generator oscillator Lie algebra $\mathcal{H}_4 = \mathrm{span}\{H, Q, P, \hat{1}\}$ of quantum mechanics (containing the Heisenberg algebra $\mathcal{H}_3 = \mathrm{span}\{Q, P, \hat{1}\}$). In one previous work [9], with the purpose of endowing Q with a finite set of position eigenvalues $x \in \{-j, -j+1, \ldots, j\}$ (we write $x\big|_{-j}^{j}$), the basic commutator was taken to be $[Q, P] = \mathrm{i}\left[H - \left(j + \frac{1}{2}\right)\hat{1}\right] =: \mathrm{i}J_3$, in a matrix representation of the Lie algebra $\mathrm{so}(3) = \mathrm{su}(2) = \mathrm{span}\{Q, P, J_3\}$ of spin j, so the three operators have the same finite spectrum $x\big|_{-j}^{j}$. This model, called the su(2) oscillator, has been applied to study the parallel processing of finite signals and pixellated images [10]. The general conditions to include oscillator dynamics in associative algebras were given in [7].

In a previous paper of the same authors [8], a *q*-algebraic associative structure was proposed on the basis of the quantum algebra $\mathrm{su}_q(2)$, the Hamiltonian having a lower-bound equally-spaced spectrum. Using a non-standard basis to define position and momentum operators that allowed analytic expressions, their spectra were determined to be a finite set of non-equally spaced points $x_s = \frac{1}{2}\sinh(s\kappa)/\sinh\frac{1}{2}\kappa$, with $s\big|_{-j}^{j}$ and $q = \mathrm{e}^{-\kappa} \in (0, 1)$. Also, explicit expressions were obtained for the wavefunctions in position and momentum, in terms of the dual *q*-Kravchuk polynomials, related by a fractional finite Fourier–*q*-Kravchuk matrix transform, and a natural representation in a *sui generis* phase space. The present paper is a continuation of the research in [11] that constructed quantum oscillators with continuous bounded spectra for the position and momentum operators.

In this paper, we build an oscillator model on the basis of the Fock representation of a quadratic associative algebra which is a *q*-deformation of the standard oscillator algebra; we

denote this by $\mathcal{D}_q\mathcal{H}_4$, which will be defined in section 2, while the physical interpretation of the participant operators that characterizes this model are set forth in section 3. The spectrum of the Hamiltonian in the algebra is lower-bound and equally spaced, as in its standard counterpart. The position spectrum and wavefunctions, orthonormal under a specific scalar product over positions, are obtained explicitly in section 4, and the momentum wavefunctions in section 5. This model can be characterized for having a space of positions given by an infinite non-degenerate point set contained in the interval $[-1, 1]$. The coordinate and momentum realizations of the oscillator are given in sections 6 and 7; in terms of these, we can write the harmonic oscillator motion in section 8, with a summation kernel is a new, which is a (denumerable infinite) relative of the fractional Fourier transform of the standard case. Concluding remarks are offered in section 9.

We use the notations that are standard in the theory of basic hypergeometric functions and q-orthogonal polynomials (see, e.g., [12]), and we assume throughout that q is a fixed real number in $(0, 1)$.

2. The quadratic algebra $\mathcal{D}_q\mathcal{H}_4$

We define the algebra $\mathcal{D}_q\mathcal{H}_4$ as the associative algebra generated by a vector basis of elements I_+, I_-, I_0, satisfying the following commutation relations:

$$[I_0, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = q^{I_0} - (1+q)q^{2I_0}. \tag{4}$$

Equivalently, introducing $I_1 := I_+ + I_-$ and $I_2 := \mathrm{i}(I_+ - I_-)$, we can characterize this algebra by

$$[I_0, I_1] = -\mathrm{i}I_2, \qquad [I_0, I_2] = \mathrm{i}I_1, \qquad [I_1, I_2] = -2\mathrm{i}(q^{I_0} - (1+q)q^{2I_0}). \tag{5}$$

The first relation in (4) can be written in the form

$$q^{I_0} I_\pm q^{-I_0} = q^{\pm 1} I_\pm. \tag{6}$$

This relation and the fact that both q^{I_0} and q^{2I_0} appear in the second relation of (4) shows that $\mathcal{D}_q\mathcal{H}_4 = \mathrm{span}\{I_1, I_2, q^{I_0}, q^{2I_0}\}$ is a quadratic associative algebra. This is a q-deformation of the oscillator algebra $\mathcal{H}_4$ because $\lim_{q\to 1^-}\mathcal{D}_q\mathcal{H}_4 = \mathcal{H}_4$. Indeed, in the limit $\lim_{q\to 1^-}$, we obtain from (5) the relations

$$[I_0, I_1] = -\mathrm{i}I_2, \qquad [I_0, I_2] = \mathrm{i}I_1, \qquad [I_1, I_2] = 2\mathrm{i}, \tag{7}$$

which are equivalent to the defining relations of $\mathcal{H}_4$.

We are interested in the Fock representation of the algebra $\mathcal{D}_q\mathcal{H}_4$; this is an irreducible representation constructed on a Hilbert space with the orthonormal basis of vectors $e_n, n \in \{0, 1, 2, \ldots\}$ (i.e., $n|_0^\infty$). In this representation, using the 'box' q-number $[a]_q := (1-q^a)/(1-q)$, the operators of the algebra act by raising and lowering the number n of e_n,

$$I_+ e_n = \sqrt{q^{n+1}[n+1]_q}\, e_{n+1}, \qquad I_- e_n = \sqrt{q^n[n]_q}\, e_{n-1}, \tag{8}$$

$$I_0 e_n = n e_n, \qquad \text{i.e.,} \quad q^{I_0} e_n = q^n e_n, \tag{9}$$

and the hermiticity conditions $I_+^* = I_-$ and $I_0^* = I_0$ are satisfied.

In order to have a functional realization of this representation, we consider the space $\mathcal{P}$ of all polynomials in one supplementary variable y, and introduce its basis of monomials

$$e_n \leftrightarrow e_n(y) := c_n y^n, \qquad c_n = \frac{q^{n(n-1)/4}}{(q; q)_n^{1/2}}, \qquad n|_0^\infty, \tag{10}$$

where $(a; q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1})$ and $(a; q)_0 = 1$. Acting on analytic functions $f(y) \in \mathcal{P}$, the Fock representation can be written in terms of the scale T_a and q-difference D_q operators,

$$I_+ = \sqrt{\frac{q}{1-q}} y T_q, \qquad q^{I_0} = T_q, \qquad I_- = [q(1-q)]^{1/2} D_q; \tag{11}$$

$$T_a f(y) = f(ay), \qquad D_q f(y) = \frac{f(y) - f(qy)}{1 - q}. \tag{12}$$

This realization of the algebra is equivalent to that in (8) and (9), with the functions $e_n(y)$ playing the role of the basis elements e_n as eigenfunctions of the weight operator I_0.

Let us now introduce a scalar product into the space of polynomials $\mathcal{P}$. This scalar product is of a Fisher-type scalar product and is given by the formula

$$\langle f_1(y), f_2(y) \rangle = f_1(\widetilde{D}_q) f_2^\dagger(y)|_{y=0},$$

where $\widetilde{D}_q := (1 - q) T_{q^{-1}} D_q$, f_1 and f_2 are polynomials, and $f_2^\dagger$ denotes the polynomial f_2, whose coefficients are replaced by their complex conjugate ones. In this formula, we have the action of the difference operator upon the polynomial $f_2^\dagger$. Then it is directly verified that

$$\langle e_n, e_{n'} \rangle = \delta_{n,n'}, \qquad n, n'|_0^\infty. \tag{13}$$

Closing the space $\mathcal{P}$ with respect to this scalar product, we obtain a Hilbert space that we denote by $\mathfrak{H}$. The space $\mathfrak{H}$ consists of functions

$$f(y) = \sum_{n=0}^{\infty} b_n e_n(y) = \sum_{n=0}^{\infty} b_n c_n y^n = \sum_{n=0}^{\infty} a_n y^n, \tag{14}$$

where $a_n = b_n c_n$, and c_n are determined by (10). Since $\langle e_n, e_{n'} \rangle = \delta_{n,n'}$ by definition, for $f_1(y) = \sum_{n=0}^{\infty} a_n y^n$ and $f_2(y) = \sum_{n=0}^{\infty} a'_n y^n$ we have

$$\langle f_1, f_2 \rangle = \sum_{n=0}^{\infty} \frac{a_n a'_n}{|c_n|^2}. \tag{15}$$

This means that the Hilbert space $\mathfrak{H}$ consists of analytic functions $f_1(y) = \sum_{n=0}^{\infty} a_n y^n$ such that

$$\|f\|^2 := \sum_{n=0}^{\infty} \left| \frac{a_n}{c_n} \right|^2 < \infty. \tag{16}$$

3. Assignment of observables to generators

The *discrete oscillator* is a class of oscillator models that depend on the parameter $q \in (0, 1)$, and based on the Fock irreducible representation of the algebra $\mathcal{D}_q \mathcal{H}_4$, where the physical observables are assigned to the spectra of self-adjoint generators of the algebra in the following way:

$$\text{position:} \quad Q := \sqrt{(1-q)/q} \, I_1, \tag{17}$$

$$\text{momentum:} \quad P := \sqrt{(1-q)/q} \, I_2, \tag{18}$$

$$\text{Hamiltonian:} \quad H := I_0 + \tfrac{1}{2} \hat{1}. \tag{19}$$

Then, due to (5), H exhibits the Hamiltonian oscillator commutation relations (1) and (2) with Q and P, and determines the basic commutator $[Q, P]$, through

$$[H, Q] = -\mathrm{i}P, \qquad [H, P] = \mathrm{i}Q, \tag{20}$$

$$[Q, P] = 2\mathrm{i}(1 - q^{-1})[q^{H-1/2} - (1+q)q^{2H-1}] =: \mathrm{i}F(H). \tag{21}$$

The operator $F(H)$ defined in (21) commutes with the Hamiltonian H and is therefore also diagonal in the Fock basis $\{e_n\}_{n=0}^{\infty}$ in (10),

$$F(H)e_n = 2(1 - q^{-1})[q^n - (1+q)q^{2n}]e_n. \tag{22}$$

This basis of $\mathfrak{H}$ thus consists of eigenfunctions of a Hamiltonian with equally-spaced eigenvalues,

$$He_n = \left(n + \tfrac{1}{2}\right)e_n, \qquad n\big|_0^{\infty}, \tag{23}$$

coinciding with the energy spectrum of the standard quantum harmonic oscillator.

From (20), the time evolution of the discrete oscillator position and momentum operators is produced by

$$\exp(\mathrm{i}\tau H) = \mathrm{e}^{\mathrm{i}\tau/2} \exp(\mathrm{i}\tau I_0), \tag{24}$$

and results in the harmonic motion

$$\begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\tau H} \begin{pmatrix} Q \\ P \end{pmatrix} \mathrm{e}^{-\mathrm{i}\tau H} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \tag{25}$$

which for $\tau \in [0, 2\pi)$ forms a group U(1) of inner automorphisms of the pair of operators Q and P, that we interpret as rotations of a phase plane around its origin (still to be studied for this model, see [8]). The phase $\mathrm{e}^{\mathrm{i}\tau/2}$ is due to the energy $\frac{1}{2}$ of the ground state e_0, while $\exp(\mathrm{i}\tau I_0)$ is the discrete counterpart of the fractional Fourier transform for this model, to be seen in section 8.

The associative algebra $\mathcal{D}_q\mathcal{H}_4$ is a q-deformation of the standard Heisenberg–Weyl algebra, where $I_\pm$ in (8) are recognized as the raising and lowering operators

$$a_q^+ := I_+ = \frac{1}{2}\sqrt{\frac{q}{1-q}}(Q - \mathrm{i}P), \qquad a_q := I_- = \frac{1}{2}\sqrt{\frac{q}{1-q}}(Q + \mathrm{i}P). \tag{26}$$

From (4) it then follows that

$$\left[\lim_{q\to 1^-} a_q, \lim_{q\to 1^-} a_q^+\right] = \lim_{q\to 1^-}[(1+q)q^{2I_0} - q^{I_0}] = 1, \tag{27}$$

so we recover the standard oscillator. This is the place to emphasize the difference between the Macfarlane–Biedenharn q-oscillator [1–6], which is defined in terms of q-commutators $a_q a_q^+ - q a_q^+ a_q$, and our discrete oscillator, which is formulated exclusively with ordinary commutators.

4. Spectrum and eigenfunctions of the position operator

A direct calculation using (8) shows that in the Fock eigenbasis $\{e_n\}_{n=0}^{\infty}$ of the Hamiltonian H, the position operator $Q = \sqrt{q^{-1} - 1}\,I_1$ acts as

$$Qe_n = \sqrt{q^n(1 - q^{n+1})}\,e_{n+1} + \sqrt{q^{n-1}(1 - q^n)}\,e_{n-1}. \tag{28}$$

Since $|q^n(1 - q^{n+1})| \leqslant 1$ for $n\big|_0^{\infty}$, the norm $\|Q\|$ of Q does not exceed 1, and hence it is a bounded operator whose eigenvalues $\{x\}$ will lie in the real interval $[-1, 1]$.

4.1. Eigenvectors of position

To find the eigenvectors $\psi_x(y)$ and the spectrum $\{x\} = \mathcal{X}$ of the position operator Q,

$$Q\psi_x(y) = x\psi_x(y), \qquad x \in \mathcal{X}, \tag{29}$$

we represent $\psi_x(y)$ in the form of a linear combination of the monomials (10),

$$\psi_x(y) = \sum_{n=0}^{\infty} p_n(x)e_n(y), \tag{30}$$

where $p_n(x)$ are coefficients depending on the points of the spectrum $x \in \mathcal{X}$.

When we substitute the expansion (30) into the equation (29), we obtain

$$\sum_{n=0}^{\infty} p_n(x)\sqrt{q^n(1-q^{n+1})}e_{n+1} + p_n(x)\sqrt{q^{n-1}(1-q^n)}e_{n-1} = x\sum_{n=0}^{\infty} p_n(x)e_n. \tag{31}$$

From here we find the following three-term recurrence relation for the coefficients $p_n(x)$ in (30),

$$xp_n(x) = \sqrt{q^n(1-q^{n+1})}p_{n+1}(x) + \sqrt{q^{n-1}(1-q^n)}p_{n-1}(x), \tag{32}$$

starting with $p_{-1}(x) = 0$ and $p_0(x) := 1$ setting the common constant factor.

We see from (32) that the coefficients $p_n(x)$ in (30) are polynomials in x of degree n, which can be evaluated uniquely. To solve the recurrence relation, we make the substitution

$$p_n(x) = (q;q)_n^{-1/2}q^{-n(n-1)/4}\widetilde{p}_n(x), \tag{33}$$

which turns (32) into

$$x\widetilde{p}_n(x) = \widetilde{p}_{n+1}(x) + q^{n-1}(1-q^n)\widetilde{p}_{n-1}(x). \tag{34}$$

Comparing this with the recurrence relation for the discrete q-Hermite polynomials of type I, given by $_2\phi_1$ basic hypergeometric polynomials [13, equation (3.28.3)],

$$h_n(z;q) := q^{n(n-1)/2}{}_2\phi_1(q^{-n}, z^{-1}; 0; q; -qz), \tag{35}$$

$$zh_n(z;q) = h_{n+1}(z;q) + q^{n-1}(1-q^n)h_{n-1}(z;q), \tag{36}$$

we establish that $\widetilde{p}_n(x) = h_n(x;q)$. We can thus write the coefficient polynomials in (30) as

$$p_n(x) = (q;q)_n^{-1/2}q^{-n(n-1)/4}h_n(x;q). \tag{37}$$

From (32) follows that these polynomials have definite parity: $p_n(x) = (-1)^n p_n(x)$.

Collecting these results we write the eigenfunctions $\psi_x(y)$ of the position operator Q as

$$\psi_x(y) = \sum_{n=0}^{\infty}(q;q)_n^{-1/2}q^{-n(n-1)/4}h_n(x;q)e_n(y) \tag{38}$$

$$= \sum_{n=0}^{\infty}(q;q)_n^{-1}h_n(x;q)y^n, \tag{39}$$

$$= \frac{(y^2;q^2)_\infty}{(xy;q)_\infty}, \tag{40}$$

where in the last expression we use the symbol $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$ and the summation formula in [13, equation (3.28.11)]. Because of the convergence of $(y^2;q^2)_\infty$ in (40) for the basis $\{\psi_x(y)\}_{x \in \mathcal{X}}$, we must restrict the domain of definition of functions $f(y) \in \mathfrak{H}$ to the open disk $|y| < 1$. Then the condition $(xy;q)_\infty < 1$ is fulfilled automatically since we saw that the eigenvalues $x \in \mathcal{X}$ of Q are contained in the interval $[-1, 1]$.

4.2. The spectrum of position

The spectrum of the self-adjoint position operator $Q \sim I_1 = I_+ + I_-$ can be found from the series (38); from (28) we see that in the basis $\{e_n(y)\}_{n=0}^{\infty}$ the operator Q is a self-adjoint Jacobi tridiagonal matrix of the form

$$\mathbf{Q} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad a_n \neq 0. \tag{41}$$

We can now use the theory of these matrices from [14, Chap. VII], (see also [6]) to connect their spectra with the corresponding measures for orthogonal polynomials. In this vein, we note that in the Fock basis the position eigenfunctions $\psi_x(y)$ are expanded in terms of the basis elements $\{e_n(y)\}_0^{\infty}$ with the *polynomial* coefficients $p_n(x)$ in (30), which are given in terms of discrete q-Hermite polynomials of type I in (37). According to the results in [14, Chap. VII], these polynomials are then orthogonal with respect to a spectral measure $\mathrm{d}\mu(x)$ of the operator, which is unique up to a constant factor, on a set $\mathcal{X} \subset \Re$ that is the simple spectrum of Q.

In finding the spectrum of the position operator Q, we recall that the discrete q-Hermite polynomials $h_n(x; q)$ obey the orthogonality relation

$$\int_{-1}^{1} (q^2 x^2; q^2)_{\infty} h_k(x; q) h_m(x; q) \, \mathrm{d}_q x = \delta_{k,m} (1-q)(q^2; q^2)_{\infty} (-1; q)_{\infty} (q; q)_m q^{m(m-1)/2}$$

$$= 2\delta_{k,m} (1-q)(q; q)_{\infty} (-q; q)_{\infty}^2 (q; q)_m q^{m(m-1)/2}, \tag{42}$$

where $\int_{-1}^{1} f(x) \, \mathrm{d}_q x$ is the symbol of the q-integral (see [13, equation (3.28.2)]). This orthogonality relation can be written in the form of a sum [12],

$$\sum_{n=0}^{\infty} (q^{2n+2}; q^2)_{\infty} q^n (h_k(q^n; q) h_m(q^n; q) + h_k(-q^n; q) h_m(-q^n; q))$$

$$= 2\delta_{k,m} (q; q)_{\infty} (-q; q)_{\infty}^2 (q; q)_m q^{m(m-1)/2}. \tag{43}$$

This means that the spectrum $\mathcal{X}$ of Q is the simple set of points

$$\mathcal{X} = \left\{ q^n, -q^n; n \big|_0^{\infty} \right\}, \tag{44}$$

and that the corresponding eigenfunctions are

$$\psi_{q^n}(y), \quad \psi_{-q^n}(y), \qquad n \big|_0^{\infty}, \tag{45}$$

given by (38)–(40). The spectrum of Q is discrete, which means that the eigenfunctions $\psi_{\pm q^n}(y)$ form a denumerable orthogonal basis in the Hilbert space $\mathfrak{H}$; we note that $\mathcal{X} \subset [-1, 1]$ has a unique accumulation point 0 that does not belong to the set.

4.3. Normalization of the eigenfunctions

The eigenfunctions of Q were determined only up to constant factors, so we proceed to normalize the eigenfunctions $\{\psi_{\pm q^s}(y)\}_0^{\infty}$ in their form (39). From (38) and the orthogonality of the basis $\{e_n(y)\}_{n=0}^{\infty}$ we obtain

$$\langle \psi_x(y), \psi_{x'}(y) \rangle_{\mathfrak{H}} = \delta_{x,x'} \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{(q; q)_n} h_n(x; q) h_n(x'; q), \tag{46}$$

where x and x' take values in $\mathcal{X} = \left\{ q^s, -q^s; s \big|_0^\infty \right\}$. We can calculate this sum as follows: we build the functions

$$\widetilde{h}_n(q^s; q) := \sqrt{\frac{(q^{2s+2}; q^2)_\infty q^s}{2(q; q)_\infty (-q; q)_\infty^2 (q; q)_n q^{n(n-1)/2}}} \, h_n(q^s; q), \tag{47}$$

$$\widetilde{h}_n(-q^s; q) := \sqrt{\frac{(q^{2s+2}; q^2)_\infty q^s}{2(q; q)_\infty (-q; q)_\infty^2 (q; q)_n q^{n(n-1)/2}}} \, h_n(-q^s; q). \tag{48}$$

We can see the $\widetilde{h}_n(\pm q^s; q)$ as the elements $(n, \pm s)$ of a matrix of numbers (integer $n \geqslant 0$ numbering rows and $s \geqslant 0$ numbering columns), written as

$$\begin{pmatrix} \{\widetilde{h}_n(q^s; q)\}_{n,s=0}^\infty \\ \{\widetilde{h}_n(-q^s; q)\}_{n,s=0}^\infty \end{pmatrix}. \tag{49}$$

Columns of this matrix are orthonormal due to the orthogonality relation (43) for the discrete q-Hermite polynomials $h_k(z; q)$. In the infinite-dimensional case, the orthonormality of columns does not immediately lead to the orthonormality of rows. But in accordance with the reasoning of [15], one can state that rows of this matrix are also orthogonal, i.e.,

$$\sum_{n=0}^\infty \widetilde{h}_n(q^s; q) \widetilde{h}_n(q^{s'}; q) = \delta_{s,s'}, \tag{50}$$

$$\sum_{n=0}^\infty \widetilde{h}_n(q^s; q) \widetilde{h}_n(-q^{s'}; q) = 0, \tag{51}$$

$$\sum_{n=0}^\infty \widetilde{h}_n(-q^s; q) \widetilde{h}_n(-q^{s'}; q) = \delta_{s,s'}. \tag{52}$$

Substituting (47) and (48) into (50)–(52), we obtain

$$\frac{(q^{2s+2}; q^2)_\infty q^s}{2(q; q)_\infty (-q; q)_\infty^2} \sum_{n=0}^\infty \frac{h_n(\pm q^s; q) h_n(\pm q^{s'}; q)}{(q; q)_n q^{n(n-1)/2}} = \delta_{s,s'}, \tag{53}$$

where one has to take only the upper or only the lower signs. Returning to the scalar product in (46), we find

$$\langle \psi_{\pm q^s}(y), \psi_{\pm q^{s'}}(y) \rangle_{\mathfrak{H}} = \delta_{s,s'} \frac{2(q; q)_\infty (-q; q)_\infty^2}{q^s (q^{2s+2}; q^2)_\infty}. \tag{54}$$

We thus arrive at the functions

$$\Psi_x(y) \equiv \Psi_{\pm q^s}(y) := \sqrt{\frac{(q^{2s+2}; q^2)_\infty q^s}{2(q; q)_\infty (-q; q)_\infty^2}} \, \psi_{\pm q^s}(y), \tag{55}$$

which are orthonormal under the scalar product (46) in $\mathfrak{H}$,

$$\langle \Psi_x(y), \Psi_{x'}(y) \rangle_{\mathfrak{H}} = \delta_{x,x'}, \qquad x, x' \in \mathcal{X}. \tag{56}$$

5. Spectrum and eigenfunctions of the momentum operator

The momentum operator $P \sim I_2 = \mathrm{i}(I_+ - I_-)$ acts on the basis $\{e_n(y)\}_0^\infty$ as

$$Pe_n = \mathrm{i}(\sqrt{q^n(1-q^{n+1})}e_{n+1} - \sqrt{q^{n-1}(1-q^n)}e_{n-1}) \tag{57}$$

[cf (28)]. When we change this basis to another $\{\widetilde{e}_n\}_0^\infty$ with $\widetilde{e}_n = \mathrm{i}^n e_n$, one can see that in the new basis the momentum operator P acts as a matrix with the same coefficient elements as the position operator in section 4 on the former basis. This means that the spectrum of momentum P coincides with the spectrum of position Q, namely, $\mathrm{Spec}\, P = \mathcal{X}$, where $\mathcal{X}$ is given in (44). Similarly, the eigenfunctions of momentum P can be found in the same way as the eigenfunctions of Q, by using the basis $\{\widetilde{e}_n\}_0^\infty$.

Let $\phi_p(y)$ satisfy $P\phi_p(y) = p\phi_p(y)$, an eigenfunction of P corresponding to the eigenvalue p, with an expansion in the mode eigenbasis $\{e_n(y)\}_0^\infty$ given by

$$\phi_p(y) = \sum_{n=0}^\infty g_n(p)e_n(y), \tag{58}$$

where $g_n(p)$ are coefficients depending on the momentum $p \in \mathcal{X}$. Repeating the process of the previous section, one derives a three-term recurrence relation for the polynomials $g_n(p)$ and concludes that

$$g_n(p) = \mathrm{i}^n p_n(p) = \frac{\mathrm{i}^n h_n(p;q)}{(q;q)_n^{1/2} q^{n(n-1)/4}}, \tag{59}$$

where $h_n(z;q)$ are the discrete q-Hermite polynomials of type I from section 4. Hence, the eigenfunctions of momentum are

$$\phi_p(y) = \sum_{n=0}^\infty \frac{\mathrm{i}^n h_n(p;q)}{(q;q)_n^{1/2} q^{n(n-1)/4}} e_n(y) \tag{60}$$

$$= \sum_{n=0}^\infty \frac{(\mathrm{i}y)^n}{(q;q)_n} h_n(p;q) \tag{61}$$

$$= \frac{(y^2;q)_\infty}{(\mathrm{i}yp;q)_\infty}, \qquad p \in \mathcal{X} = \left\{ q^s, -q^s; s \big|_0^\infty \right\}. \tag{62}$$

To find the last two expressions we have used the same method as in the case of eigenfunctions of position in (38)–(40).

The normalized eigenfunctions of P are

$$\Phi_p(y) \equiv \Phi_{\pm q^s}(y) = \sqrt{\frac{(q^{2s+2};q^2)_\infty q^s}{2(q;q)_\infty(-q;q)_\infty^2}} \phi_{\pm q^s}(y), \tag{63}$$

satisfying $\langle \Phi_x(y), \Phi_{x'}(y) \rangle_{\mathfrak{H}} = \delta_{x,x'}, x, x' \in \mathcal{X}$.

6. Coordinate realization of the discrete oscillator

In section 3, we constructed a realization of the discrete oscillator on the space of analytic functions in the supplementary variable y with the assignment (10). It is natural to look for a realization of the oscillator on the space of functions in the position coordinate $x \in \mathcal{X}$.

Let $L^2(\mathcal{X})$ be the Hilbert space of square-summable functions over $x \in \mathcal{X}$ (the set of positions of the discrete oscillator), with the scalar product

$$\langle f_1, f_2 \rangle_{L^2(\mathcal{X})} := \frac{1}{(q^2; q^2)_\infty (-1; q)_\infty} \sum_{n=0}^\infty (q^{2n+2}; q^2)_\infty q^n (f_1(q^n) f_2^*(q^n) + f_1(-q^n) f_2^*(-q^n)), \tag{64}$$

where $*$ stands for complex conjugation.

Since the discrete q-Hermite polynomials are associated with the determinate moment problem (see, e.g., [6] for the description of this association), the set of polynomials $\{p_n(x)\}_0^\infty$ in (37) constitutes a complete set of orthonormal functions in the Hilbert space $L^2(\mathcal{X})$.

We construct a one-to-one linear isometry Ω from the Hilbert space $\mathfrak{H}$, onto the Hilbert space $L^2(\mathcal{X})$, given by

$$\Omega : \mathfrak{H} \ni e(y) \to f(x) = \langle e(y), \psi_x(y) \rangle_{\mathfrak{H}} \in L^2(\mathcal{X}), \tag{65}$$

where $\psi_x(y)$ are eigenfunctions (40) of Q. It follows from (38) that

$$\mathfrak{H} \ni e_n(y) \to \langle e_n(y), \psi_x(y) \rangle_{\mathfrak{H}} = p_n(x). \tag{66}$$

That is, Ω maps the basis $\{e_n(y)\}$ of $\mathfrak{H}$, which is orthonormal under the scalar product (54), onto the basis $\{p_n(x)\}$ of $L^2(\mathcal{X})$, which is orthonormal under (64); this means that Ω is a one-to-one isometry.

In $L^2(\mathcal{X})$, the operator Q acts through multiplication,

$$Qf(x) = xf(x). \tag{67}$$

Indeed, since $Q\psi_x(y) = x\psi_x(y)$ for $\Omega e(y) = f(x) = \langle e(y), \psi_x(y) \rangle_{\mathfrak{H}}$, we have

$$\begin{aligned} \Omega : Qe(y) \to Qf(x) &= \langle Qe(y), \psi_x(y) \rangle_{\mathfrak{H}} \\ &= \langle e(y), Q\psi_x(y) \rangle_{\mathfrak{H}} = \langle e(y), x\psi_x(y) \rangle_{\mathfrak{H}} = xf(x). \end{aligned} \tag{68}$$

We can find the action of Q, P and H on the basis elements $\{p_n(x)\}_{n=0}^\infty$ of the Hilbert space $L^2(\mathcal{X})$. According to the recurrence relation (32), which follows from the recurrence relation for the discrete q-Hermite polynomials $h_n(z; q)$, we have for the position operator Q that

$$Qp_n(x) = \sqrt{q^n(1 - q^{n+1})} \, p_{n+1}(x) + \sqrt{q^{n-1}(1 - q^n)} \, p_{n-1}(x). \tag{69}$$

It follows from formulae (3.28.7) and (3.28.8) in [13] that the momentum operator P acts on the Hilbert space $L^2(\mathcal{X})$ through

$$P = -\mathrm{i}(1 - q)q^{H-1/2} \left(D_q + \frac{1}{q^2(q^2 x^2; q^2)_\infty} D_{q^{-1}}(q^2 x^2; q^2)_\infty \right), \tag{70}$$

where $(q^2 x^2; q^2)_\infty$ is the multiplier in the orthogonality measure in the scalar product (64). In particular, P acts on the basis functions $\{p_n(x)\}_{n=0}^\infty$ as

$$Pp_n(x) = \mathrm{i}\sqrt{q^n(1 - q^{n+1})} \, p_{n+1}(x) - \mathrm{i}\sqrt{q^{n-1}(1 - q^n)} \, p_{n-1}(x). \tag{71}$$

Finally, the Hamiltonian H acts on the basis polynomials $p_n(x)$ of the Hilbert space $L^2(\mathcal{X})$ as

$$Hp_n(x) = \left(n + \tfrac{1}{2} \right) p_n(x). \tag{72}$$

Indeed, according to (23) and (66) we have

$$\begin{aligned} Hp_n(x) &= \langle He_n(y), \psi_x(y) \rangle_{\mathfrak{H}} \\ &= \left(n + \tfrac{1}{2} \right) \langle e_n(y), \psi_x(y) \rangle_{\mathfrak{H}} = \left(n + \tfrac{1}{2} \right) p_n(x). \end{aligned} \tag{73}$$
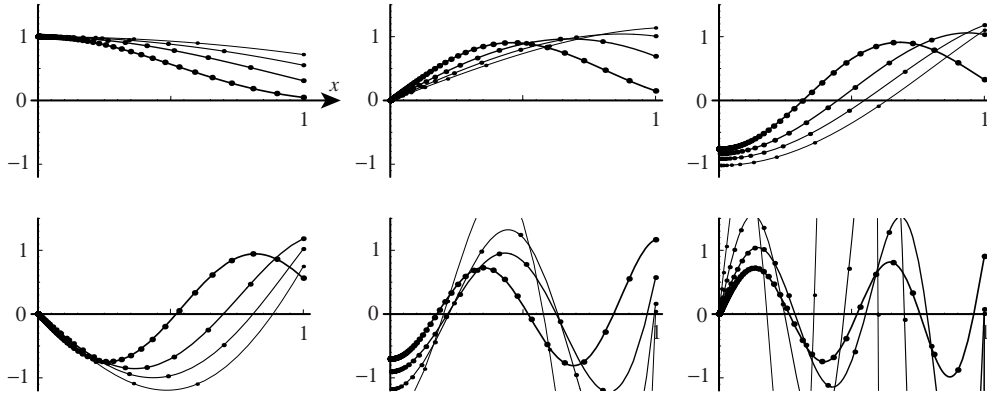
Figure 1. Discrete oscillator wavefunctions $p_n(x)$ in (37) normalized according to (90) (below), for the modes $n = 0, 1, 2, 3, 6$ and 9. The modes have definite parity, $p_n(-x) = (-1)^n p_n(x)$, so we show only the range $0 < x \leqslant 1$. Continuous lines correspond to the functions of continuous x, while the dots indicate the values of the function on the points $\mathcal{X}(q) \supset \{q^n; n|_0^\infty\}$. Each graph includes four values of q: 0.6, 0.7, 0.8 and 0.9, indicated by lines and dots of increasing thickness.

In figure 1, we show some of the lowest discrete oscillator modes $p_n(x)$ in (37) and (72), both as continuous functions of $x \in (0, 1]$ and their values at the orthogonality set $\mathcal{X}$, for various values of q. While the continuous functions exhibit strong oscillations (increasing with n and $1/q$), their values on $\mathcal{X}(q)$ remain well bounded within the range in the figure. As $q \to 1^-$, the corresponding points in $\mathcal{X}$ densify, evincing the resemblance of the discrete oscillator with the standard oscillator wavefunctions. This form of convergence should be studied further, since a change of scale appears necessary as well as a discrete measure that becomes a continuous Riemann integral in the limit. One analogue for this limit appears in [16, figure 4], where the Meixner functions that describe the discrete model converge to the Laguerre–Gauss modes of a radial oscillator; in that case though, the limit is from hyperboloids to the cone in the three-dimensional space of the Lie algebra su(1, 1), and is not a q-deformation.

7. Momentum realization of the discrete oscillator

Consider the Hilbert space $L^2(\mathcal{P})$ of square-integrable functions $f(p)$ in the momentum coordinate p in the oscillator with the same scalar product as in (64), where $\mathcal{P} = \mathcal{X}$ is the spectrum of the momentum operator P, coinciding with the spectrum of Q. The coefficient polynomials $g_n(x)$ in formula (59) for the eigenfunctions of momentum, $\phi_p(y)$ in (58), constitute an orthonormal basis in $L^2(\mathcal{P})$.

To formalize this consideration, we construct, as in the previous section, a one-to-one linear isometry $\widetilde{\Omega}$ from the Hilbert space $\mathfrak{H}$ onto the Hilbert space $L^2(\mathcal{P})$, given by

$$\widetilde{\Omega} : \mathfrak{H} \ni e(y) \to f(p) := \langle e(y), \phi_p(y) \rangle_{\mathfrak{H}} \in L^2(\mathcal{P}), \tag{74}$$

where $\phi_p(y)$ are the eigenfunctions of momentum P in (60)–(62). [Compare with (65) requiring the position eigenfunctions $\psi_x(y)$ in (38)–(40).] From here it is evident that

$$\mathfrak{H} \ni e_n(y) \to \langle e_n(y), \phi_p(y) \rangle_{\mathfrak{H}} = g_n(p), \tag{75}$$

that is, $\widetilde{\Omega}$ is a one-to-one isometry and maps the orthonormal basis $\{e_n(y)\}_0^\infty \in \mathfrak{H}$ onto the orthonormal basis $\{g_n(p)\}_0^\infty \in L^2(\mathcal{P})$.

The momentum operator P acts on $L^2(\mathcal{P})$ as a multiplication operator on all functions of p,

$$Pg(p) = pg(p). \tag{76}$$

The action of Q, P and H on the basis of polynomials $g_n(p)$ can be found in the form of the recurrence relations

$$Qg_n(p) = \sqrt{q^n(1-q^{n+1})}g_{n+1}(p) + \sqrt{q^{n-1}(1-q^n)}g_{n-1}(p), \tag{77}$$

$$Pg_n(p) = i\sqrt{q^n(1-q^{n+1})}g_{n+1}(p) - i\sqrt{q^{n-1}(1-q^n)}g_{n-1}(p), \tag{78}$$

$$Hg_n(p) = \left(n + \tfrac{1}{2}\right)g_n(p). \tag{79}$$

8. Harmonic evolution in position space

According to (24) and (25), the action of the operator $\exp(i\tau H)$ is the time evolution of the discrete oscillator. On the basis (10) of functions $\{e(y)\}_0^\infty$ that is orthonormal with respect to the scalar product (13), this action is

$$e^{i\tau H}e_n(y) = e^{i\tau/2}\,e^{in\tau}e_n(y) = e^{i(n+1/2)\tau}e_n(y). \tag{80}$$

The operator $\exp(i\tau H)$ also acts on the Hilbert space $L^2(\mathcal{X})$, which is characterized by the scalar product (64). Now consider the isometry between these two spaces,

$$\mathfrak{H} \ni e(y) \rightarrow f(x) := \langle e(y), \psi_x(y)\rangle_{\mathfrak{H}} \in L^2(\mathcal{X}), \tag{81}$$

that maps functions $e(y)$ onto functions $f(x)$ of the discrete position coordinate $x \in \{-q^n, q^n\}_0^\infty = \mathcal{X}$. Then to $\exp(i\tau H)e(y) \in \mathfrak{H}$ there corresponds a function $\exp(i\tau H)f(\pm q^s)$ of $x = \pm q^s$,

$$e^{i\tau H}f(\pm q^s) = \langle e^{i\tau H}e(y), \psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} = \langle e(y), e^{-i\tau H}\psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} \tag{82}$$

$$= \sum_{n=0}^\infty \langle e(y), e_n\rangle_{\mathfrak{H}}\langle e_n, e^{-i\tau H}\psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} \tag{83}$$

$$= \sum_{n=0}^\infty \langle e(y), e_n\rangle_{\mathfrak{H}}\langle e^{i\tau H}e_n, \psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} \tag{84}$$

$$= \sum_{n=0}^\infty\sum_{m=0}^\infty(\langle e(y), \Psi_{q^m}(y)\rangle_{\mathfrak{H}}\langle\Psi_{q^m}(y), e_n\rangle_{\mathfrak{H}} + \langle e(y), \Psi_{-q^m}(y)\rangle_{\mathfrak{H}}\langle\Psi_{-q^m}(y), e_n\rangle_{\mathfrak{H}})$$

$$\times e^{i\tau(n+1/2)}\langle e_n, \psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} \tag{85}$$

$$= \sum_{m=0}^\infty(K^\tau(\pm q^s, q^m)f(q^m) + K^\tau(\pm q^s, -q^m)f(-q^m)); \tag{86}$$

where, according to (38),

$$K^\tau(\pm q^s, \pm q^m) = c_s\sum_{n=0}^\infty\langle\psi_{\pm q^m}(y), e_n\rangle_{\mathfrak{H}}\,e^{i\tau(n+1/2)}\langle e_n, \psi_{\pm q^s}(y)\rangle_{\mathfrak{H}} \tag{87}$$

$$= c_s\,e^{i\tau/2}\sum_{n=0}^\infty\frac{q^{-n(n-1)/2}}{(q;q)_n}h_n(\pm q^m; q)\,e^{in\tau}h_n(\pm q^s; q) \tag{88}$$

and

$$c_s := \frac{q^s(q^{2s+2}; q^2)_\infty}{2(q; q)_\infty(-q; q)_\infty^2}. \tag{89}$$

Being elements of $L^2(\mathcal{X})$, the functions $f(x)$, $x \in \mathcal{X}$, enter into the scalar product (64) as a sum with the weight function $(x^2q^2; q^2)_\infty$. Because we intend to interpret these as wavefunctions of a quantum oscillator, and display them for comparison with their standard shapes, we should absorb this weight into the functions,

$$F(x) = \sqrt{(x^2q^2; q^2)_\infty}\, f(x), \tag{90}$$

so that their scalar product acquires, from (64), the 'standard' form,

$$\langle F_1, F_2 \rangle_\mathcal{X} := \frac{1}{(q^2; q^2)_\infty(-1; q)_\infty} \sum_{x \in \mathcal{X}} |x| F_1(x) F_2(x)^*, \tag{91}$$
$$x \equiv x(\pm, n) = \pm q^n \in \mathcal{X}, \quad n|_0^\infty.$$

The matrix elements of operators will be correspondingly rescaled by (90).

The oscillator evolution (82)–(86) can be used to define the fractional discrete Fourier transform on the functions $F(x)$, $x \in \mathcal{X}$, rescaled as in (91). We note that the fractional Fourier integral transform for angle τ differs from the standard harmonic oscillator evolution by a phase $e^{i\tau/2}$ that is due to the ground energy $\frac{1}{2}$ in the oscillator, so the fractional Fourier transform is

$$\Phi(\tau) := e^{-i\tau/2} \exp(i\tau H). \tag{92}$$

Its action on the rescaled discrete wavefunctions will have the form

$$\Phi(\tau)F(x) = \sum_{x' \in \mathcal{X}} \Phi(x, x'; \tau)F(x'), \tag{93}$$

$$\Phi(x, x'; \tau) = e^{-i\tau/2} \sqrt{\frac{(x^2q^2; q^2)_\infty}{(x'^2q^2; q^2)_\infty}} K^\tau(x, x'), \tag{94}$$

where $x, x' \in \mathcal{X}$ and $K^\tau(x, x')$ is given by (87)–(88) for $x = \pm q^s$ and $x' = \pm q^m$. Unfortunately, the bilinear generating function (88) of discrete q-Hermite polynomials of type I could not be summed to a closed form.

9. Concluding remarks

We constructed a model of the harmonic oscillator that can be realized on bases of coordinate and momentum Hilbert spaces, and its energy modes expressed in terms of discrete q-Hermite polynomials of type I. The spectrum of the Hamiltonian coincides with that of the standard harmonic oscillator in quantum mechanics, while the position and momentum operators in this model have discrete, denumerably infinite spectra that depend on the extension parameter q contained in the interval $[-1, 1]$.

Contrary to other models that use discrete q-Hermite polynomials [17–19], our models (the present one and that in [8]) fulfil the basic Hamilton equations in the form $[H, Q] = -iP$ and $[H, P] = iQ$, with *standard commutators*—and not q-commutators [1, 2]. Because of this important circumstance, the time evolution of the model is a Lie group, which but for a phase is that of fractional Fourier transforms associated with this model [20]. This discrete oscillator is a new and non-trivial deformation of the standard quantum harmonic oscillator;

it allows the extension of other standard concepts of phase space, such as coherent states [8], that will be examined elsewhere.

We believe that the discrete oscillator model can appropriately describe discrete quantum systems on bounded point lattices, and also contribute significantly to the general theory of special functions.

Acknowledgments

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